Exactness of the reduced crossed product functor for arithmetic groups

A locally compact group G is called **exact** (or C*-exact) if for any short exact sequence of C*-algebras

$$0 \to I \to A \to A/I \to 0$$

where each algebra is endowed with a strongly continuous G-action making the sequence equivariant, the associated sequence or reduced crossed products

$$0 \to I \rtimes_r G \to A \rtimes_r G \to (A/I) \rtimes_r G \to 0$$

is also exact. I.e. $- \rtimes_r G$ is an exact functor from the category of G-C*-algebras (with equivariant morphisms) to the category of C*-algebras.

As opposed to the full crossed product functor, which is always exact, there are groups for which the functor is not exact. For a reference, here are two pathological examples of non-exact groups

- Called by some authors the Gromov monsters, these are defined in [2] and are non-exact discrete groups. There seems to be no imbedding of these groups into some $B(H)$ for any Hilbert space H, so they are non-isomorphic to any groups that will be studied here.
- (Osajda) There are residually finite non-exact groups. Note that arithmetic groups are residually finite (proof) $G(p\mathbb{Z})$ and $G(q\mathbb{Z})$ are subgroups of $G(\mathbb{Z})$ with finite index that intersect only at I if p and q are coprimes.

It turns out that if G is any discrete group we have

$$C_r^*(G) \text{ is an exact C*-algebra} \quad \Leftrightarrow \quad - \rtimes_{r,\alpha} G \text{ is an exact functor}$$

The $\Leftarrow$ implication is always true for any locally compact group since if G acts trivial on a C*-algebra A, then $A \otimes_{min} C_r^*(G) \simeq A \rtimes_{\alpha,r} G$. This statement is proved in [1][Theorem 5.2].

Since we will be dealing with arithmetic groups, we can exploit the following theorem from [3]: Let K be a field and let n be a positive integer. The reduced group C*-algebra of every subgroup of $GL_n(K)$ is exact.

Hence if we believe the two above results we get the following

Theorem 1. *If K is any field and Γ is a discrete linear subgroup of $GL_n(K)$, the functor $- \rtimes_r \Gamma$ is exact.*

Hopefully this covers all the groups in question.

References

[1] Kirchberg, Eberhard, and Simon Wassermann. "Exact groups and continuous bundles of C*-algebras." Mathematische Annalen 315.2 (1999): 169-203.

[2] M. Gromov, *Random walk in random groups.* Geom. Funct. Anal., (1)13, 2003

[3] Guentner, Erik, Nigel Higson, and Shmuel Weinberger. "The Novikov conjecture for linear groups." Publications mathématiques de l'IHÉS 101.1 (2005): 243-268.