Classification of locally symmetric spaces of rank 1 through K-theory

Torstein Ulsnaes

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The **rank** of the symmetric space is the largest dimension of its flat totally geodesic submanifolds.
For any symmetric space X the $G = Iso(X)^0$ acts **transitively**.
If $x \in X$, $K = Stab(x)^0$ is compact subgroup in G, and

$$G/K \simeq X.$$

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Examples: $\mathbb{H}^n_{\mathbb{R}}, \mathbb{H}^n_{\mathbb{C}}, \mathbb{H}^n_{Q}$

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The **geodesic compactification** $\overline{X}$ of a symmetric space X of noncompact type is the compactification obtained by adding a sphere $\partial X := S^{n-1}$ to X $(n = dim(X))$ as follows (blackboard!) The action of G extends continuous to $\overline{X}$. Much information about the locally symmetric space $\Gamma \backslash X$ can be recovered from the dynamical system

$$(\partial X, \Gamma)$$

Mostow rigidity theorem

Assuming X, X' are symmetric space of noncompact type of rank 1, not isomorphic to $\mathbb{H}^2_{\mathbb{R}}$. Let $\Gamma \subset G$, $\Gamma' \subset G'$ be two torsion free co-compact lattices, then the following are equivalent

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Crossed product C*-algebra

Given a dynamical system (X, G), a covariant representation (π, u) of (X, G) are two representations

$$\pi : C_0(X) \to B(H) \qquad u : G \to U(B(H))$$

satisfying

$$\pi(gf) = u(g)^\star \pi(f) u(g)$$

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The **crossed product C*-algebra**, denoted $C_0(X) \rtimes G$ is the completion $C_c(G, C_0(X))$ with respect to the norm

$$||f|| = \sup_{(\pi,u)} \int_G \pi(f(g))u(g)$$

over all covariant representations (π, u) of (X, G)

Mostow rigidity and the boundary C*-algebra

Question 1: Assuming X and Γ are as in Mostows theorem, are the boundary C*-algebras

$$C(\partial X) \rtimes \Gamma$$

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Noncommutative mostow rigidity theorem.

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Question 1': Does the K-theory of $C(\partial X) \rtimes \Gamma$ uniquely determine the space $\Gamma \backslash X$ up to isometry?

Further simplifications

For Γ G as above, with a mild condition on $\Gamma\backslash G/K$ (which is, having a spin^c-structure) we have that

$$K_\star(C(\partial X) \rtimes \Gamma) = K_\star(C(\Gamma\backslash G/M))$$

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where $M = P \cap K$ and K is the maximal compact subgroup of G. We are thus lead to the following simplification of question 1
Question 1" Does the topological K-theory of $\Gamma\backslash G/M$ determine the symmetric space $\Gamma\backslash X$ up to isometry?

Some results: Comparing with classifying spaces

Theorem

For X as above, and $\dim(X)$ odd, we have

$$K(\Gamma\backslash G/M) = K_\star(\Gamma\backslash G/K) \otimes \mathbb{Z}^2$$

(Proof on the blackboard, if time...)

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On the other hand, if X is odd dimensional we have the following

Theorem

If dim(X) is odd, then

$$K^0(\Gamma\backslash G/M) = K^1(\Gamma\backslash G/M)$$

A note on torsion

Theorem

If X is a hyperbolic space with $n = dim(X)$ and Γ torsion free and cocompact lattice, then the (cohomological) betti numbers are

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Proof.

The Chern character is a rational isomorphism. If n is odd we get

$$K^\star(\Gamma\backslash G/M) \otimes \mathbb{Q} = K^\star(\Gamma\backslash G/K) \otimes \mathbb{Z}^2 \otimes \mathbb{Q} = H^\star(\Gamma\backslash G/K, \mathbb{Q}) \otimes \mathbb{Z}^2.$$

□

Reducing to maximal tori

It K is a compact group Lie group with maximal torus T_K then it is well known that

$$K_K^\star(X)^{|W_K|} = K_{T_K}^\star(X)$$

where

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$K^\star(\Gamma\backslash G/K)^{|W_K|} = K^\star(\Gamma\backslash G/M)^{|W_M|} \oplus K^\star(\Gamma\backslash G/T)$

Proof.

We have $T_K = T_m \oplus T$, where $T = T_k/T_m$ which yields

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