Compactifications of symmetric spaces and their applications

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For any symmetric space X the $G = Iso(X)^0$ acts **transitively**.
If $x \in X$, $K = Stab(x)^0$ is compact subgroup in G, and

$$G/K \simeq X.$$

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- **Non-compact type** (negative curvature)

Why study locally symmetric spaces... Number and representation theory

Let $G \subset GL_n(\mathbb{R})$ be a subgroup of finte index in the group of real points of a connected semi-simple linear algebraic group $\mathbf{G}$ over $\mathbb{R}$. $\Gamma \subset G$ a (torsion free) lattice and $X = G/K$ the associated symmetric space.

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$D(X)$ always contains the Laplacian.

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A Maass from f is called a **cusp form** if for any proper Γ-cuspidal parabolic subgroup $P = MAN \subset G$

$$\int_{(\Gamma \cap N)\backslash N} f(nx)dn = 0.$$

For cusp forms we have $f \in L^2(\Gamma\backslash X)$ and are joint eigenfunction of the algebra $D(X)$ of G-invariant differential operators on X.

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- The point spectrum of the Laplacian corresponds to the cusp forms.
- Each joint eigenspaces of $D(X)$ gives us a G-representation on $V \subset L^2(\Gamma \backslash X)$.

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It is shown in [2] that the Γ-equivariant short exact sequence

$$0 \to C_0(X) \to C(\overline{X}) \to C(\partial X) \to 0$$

induces (when Γ is exact) a six-term exact sequence

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- Statements about the group cohomology of Γ, could be inferred by this sequence.
- In some cases the C*-algebra $C(\partial X) \rtimes \Gamma$ is simple nuclear and purely infinite, possibly classifiable by Elliott invariants.
- Open question: does the Baum-Connes conjecture holds for $\Gamma = SL_n(\mathbb{Z}) \subset SL_n(\mathbb{R})$ when $n \geq 3$.

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- $SL_n(\mathbb{R})/SO_n(\mathbb{R})$ is a symmetric space of noncompact type. For the lattice $SL_n(\mathbb{Z}) \subset SL_n(\mathbb{R})$ the Baum-Connes conjecture is still open for $n \geq 3$.

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The **Karpelevic compactification is the largest** -

$$\overline{X}^K \twoheadrightarrow \overline{X}^U, \quad U \in [S, M, F, G].$$

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- Use Ellis theorem and Zorn's lemma.

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Properties preserved by projective limits

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Results so far -

- There exists a maximal compact metric G-compactification $\beta_G X$ of covering dimension $\leq n = dim(X)$.
- In the case of $rank(X) = 1$ we have $\beta_G X \simeq \overline{X}^K \simeq \overline{X}^G$.

Bibliography

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