Optimal Transport and Geometric Inequalities

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Aim
Present some applications of Optimal Transport to geometric inequalities for smooth/non-smooth manifolds.

Plan
▶ General overview on Optimal Transport
▶ Theory of Curvature-Dimension condition
▶ Functional Inequalities: Levy-Gromov isoperimetric inequality
Optimal Transport: Formulation

How to minimise total transport cost? (Monge 1781, Kantorovich 1942)

\[
\begin{align*}
\mathbb{R}^n \times \mathbb{R}^n & \quad y = T(x) \\
\int f_1(x) \, dx & = \int f_2(y) \, dy,
\end{align*}
\]

Given a cost function \( c(x, y) \), the total transport cost of \( T \) is

\[
C(T) = \int \mathbb{R}^n c(x, T(x)) f_1(x) \, dx.
\]
Optimal Transport: Formulation

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If $\int f_1 = \int f_2$, $T$ is a transport map from $f_1$ to $f_2$ iff for any $A \subset \mathbb{R}^n$

$$\int_A f_2(x) \, dx = \int_{T^{-1}(A)} f_1(x) \, dx, \quad i.e. \quad T_\#(f_1 \, dx) = f_2 \, dx,$$
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\[ T \rightarrow \int_{\mathbb{R}^n} c(x, T(x)) f_1(x) \, dx, \quad T \text{ transport map from } f_1 \text{ to } f_2. \]
Optimal Transport: Monge problem

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Main issues with the minimization problem

- \( T \) is (smooth) transport map iff \( f_2(T(x))|\det DT(x)| = f_1(x) \). Highly non-linear constrain.
- The set of transport maps is not closed in any reasonable topology.
- Replace \( f_1, f_2 \) with any \( \mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^n) \) to obtain the general Monge problem: the set of transport maps can be empty.
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\( \leadsto \) Kantorovich relaxation rewrite the total transportation cost

\[ \int_{\mathbb{R}^n} c(x, T(x)) \mu_1(dx) = \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y)((id, T) \# \mu_1)(dxdy) \]
A transport map $T$ seen as a measure on its graph $(id, T)_{#}\mu_1$ becomes a transport plan

$$\Pi(\mu_1, \mu_2) = \{\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n): (P_i)_{#}\pi = \mu_i, \ i = 1, 2\}.$$
Optimal Transport: Monge-Kantorovich problem

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Set of transport plans is weakly closed and convex. Monge-Kantorovich problem minimize the linear functional

$$\Pi(\mu_1, \mu_2) \ni \pi \mapsto \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y)\pi(dx\,dy).$$

If $c: \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is l.s.c., existence of a solution.
Structure of optimal plans is obtained via classical duality theory.
Optimal Transport: Solutions for $d^2$ and $d$

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- $X = M^n$ Riem. mfld $c = d_g^2$ (Brenier, McCann, Gangbo).

Given $\mu_1 = f_1 dvol_g$ and any $\mu_2$, $\exists!$ optimal transport map

$$T(x) = \exp_x(-\nabla \psi(x)), \quad \psi : M^n \to \mathbb{R}, \quad d_g^2 - \text{concave},$$

$$\psi^{cc} = \psi \quad \text{where} \quad \psi^c(y) = \inf_{x \in M} \frac{d_g^2(x,y)}{2} - \psi(x).$$
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- $X = M^n$ Riem. mfld $c = d_g$ (Feldman, McCann).

Given $\mu_1, \mu_2$ there exists $u : X \to \mathbb{R}$ 1-Lipschitz function so that

$$\pi \text{ optimal } \iff \pi(\{(x,y) : u(x) - u(y) = d_g(x,y)\}) = 1.$$  

Optimal path for $c = d_g$ are along steepest descent of $u$.  

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- **Analysis and PDEs**: Gradient flows, JKO scheme. Monge-Ampere equation ($c(x, y) = |x - y|^2$).

- **Physics**: Random matching problem (squared Euclidean dist.), Density Functional Theory (Coulomb cost), Einstein equation of general relativity (Lorentzian cost function).

- **Geometry of metric spaces**: new class of metric spaces by Lott-Sturm-Villani verifying $\text{Ric} \geq K$ and $\text{dim} \leq N$ in a synthetic sense, called $\text{CD}(K, N)$.

- **Data science and Economy**: Entropic regularisation (Sinkhorn, Schrodinger problem), mixed problems (Hellinger-Kantorovich).
Let \((M^n, g)\) be an \(n\)-dimensional Riemannian manifold. Denote \(\text{Sec}\) the sectional curvature and \(\text{Ric}\) the Ricci curvature.

- For \(K \in \mathbb{R}\) we write \(\text{Sec} \geq K\) (resp. \(\leq K\)) if for every \(p \in M\) and every 2-dim plane \(\Pi \subset T_pM\) it holds \(\text{Sec}_p(\Pi) \geq K\) (resp. \(\leq K\)).

- \(\text{Ric}_p : T_pM \times T_pM \rightarrow \mathbb{R}\) is a quadratic form. We write \(\text{Ric} \geq K\) (resp. \(\leq K\)) if the quadratic form \(\text{Ric}_p - Kg_p\) is non-negative (resp. non-positive) definite at every \(p \in M\).
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Examples

- $n$-dimensional euclidean space: $\text{Sec} \equiv 0$, $\text{Ric} \equiv 0$.
- $n$-dimensional round sphere of radius 1: $\text{Sec} \equiv 1$, $\text{Ric} \equiv n - 1$.
- $n$-dimensional hyperbolic space: $\text{Sec} \equiv -1$, $\text{Ric} \equiv -(n - 1)$. 
Geometry of metric spaces: basics

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Natural question \((M, g)\) smooth Riem. manifold. Assume some upper/lower bounds on \(\text{Sec}\) or on \(\text{Ric}\); what can we say on \((M, g)\)?
Basics on comparison geometry

- Upper/Lower bounds on the Sec are strong assumptions with strong implications (definition of Alexandrov spaces: non smooth spaces with upper/lower bounds on Sec).

- Upper bounds on the Ricci curvature are very (too) weak assumption for geometric conclusions. Lokhamp Theorem: any closed mfld of $\dim \geq 3$ carries a metric with negative $Ric$. 

- Lower bounds on the Ric natural framework for comparison geom.

- Bishop-Gromov volume comparison: If $Ric \geq 0$ then for all $x \in M$ $R \rightarrow vol(B_R(x))/\omega_N R^N$ is monotone non-increasing

- Laplacian comparison,

- Cheeger-Gromoll splitting,

- Levy-Gromov isoperimetric inequality

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Gromov in the '80ies:

- notion of convergence for Riemannian manifolds: Gromov-Hausdorff convergence (for non-compact manifolds, more convenient a pointed version, called pointed Gromov-Hausdorff convergence \( \sim \) GH-convergence of metric balls of every fixed radius).
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**Big Question** what about the compactification of the space of Riem. mfld with Ricci curvature bounded below (by, say, $-1$)?

**Hope** useful also to establish properties for smooth manifolds.

Non-intrinsic point of view consider the non-smooth space arising as limits of smooth objects. Dichotomy collapsing (loss of dim in the limit)-non collapsing. Very powerful for local struct. properties.

Analogy Define $W^{1,2}$ as completion of $C^\infty$ endwed with $W^{1,2}$-dist.

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**Analogy** Define $W^{1,2}$ as completion of $C^\infty$ endwed with $W^{1,2}$-dist. $W^{1,2}$ can be defined also in completely **intrinsic way** without passing via approximations (very convenient for doing calculus of variations).

**Role of OT** define in an intrisic-axiomatic way a non-smooth space with Ricci curvature bounded below by $K$ and dimension bounded above by $N$ (containing ricci limits no matter if collapsed or not).

$\implies$ Weak version of a Riemannian manifold with $\text{Ric} \geq K$. 

Optimal Transport: Cornerstone

Interplay of Optimal Transport, entropy and curvature

Ricci curvature in terms of geodesic convexity of entropy along $L_2$

Optimal Transport, $c(x,y) = d^2_{g}(x,y)$ (Lott-Villani, Sturm '06)

$\rho_0 \rho_1 / 2 \rho_1 t$

$\operatorname{Ent}(\rho) = \int \rho(x) \log \rho(x) \, dx$

Giving:

$\operatorname{Ric} \geq K$ if and only if $\operatorname{Hess} \operatorname{Ent} \geq K$.

LSV theory: new approach to non-smooth metric spaces

Examples: manifolds with $\operatorname{Ric} \geq K$, Alexandrov spaces, normed and Finsler spaces, limits of those spaces
Ricci curvature in terms of geodesic convexity of entropy along $L^2$ Optimal Transport, $c(x, y) = d_g^2(x, y)$ (Lott-Villani, Sturm '06)

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- **LSV theory**: new approach to non-smooth metric spaces

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