What is Noncommutative Geometry?

How a geometry can be commutative and why mine is not

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**Theorem**

Two smooth manifolds $M, N$ are diffeomorphic if and only if their algebras of smooth functions $\mathcal{C}^\infty(M)$ and $\mathcal{C}^\infty(N)$ are isomorphic.
Doing Geometry Without a Geometric Space

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This means that the algebra $\mathcal{C}^{\infty}(M)$ contains enough information to codify the whole geometry of the manifold:

1. Vector fields: linear derivations of $\mathcal{C}^{\infty}(M)$
2. Differential 1-forms: $\mathcal{C}^{\infty}(M)$-linear forms on vector fields
3. ...
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**Question**

- *Do we really need a manifold’s points to study it?*
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Question

- Do we really need a manifold’s points to study it?
- Do we really use the commutativity of the algebra $\mathcal{C}^\infty(M)$ to define the aforementioned objects?
Consider the following sentences:

- Darling I love you
- Leaving your idol
- Avoiding our yell
Why Consider Noncommutativity?

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They are all anagrams of the letters: adegi\textsuperscript{2} l\textsuperscript{2} no\textsuperscript{2} ruvy
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**Moral of the Story**

*Commutativity \( \Rightarrow \) Loss of Information*
Fundamental Idea of Noncommutative Geometry

Replace $C^\infty(M)$ with a **possibly** noncommutative algebra $A$ and regard $A$ as the algebra of functions on a "noncommutative smooth manifold".

\[ M \longrightarrow C^\infty(M) \longrightarrow \text{Vector Fields, Diff. Forms, ...} \]

\[ ? \longrightarrow A \longrightarrow \text{Analogous Algebraic Constructions} \]
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- Noncommutative manifolds do **NOT** exist.
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- **If** $A$ **is commutative** we want back all the known machinery.
- **Noncommutative manifolds do NOT exist.**

The previous method can be adapted to study also:

- A topological space $X$ and the algebra $C(X)$
- A measure space $(X, \mu)$ and the algebra $L^\infty(X, \mu)$
- ...
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Noncommutative Topology
**Definition**

A **C*-algebra** is a complex algebra $A$ endowed with a norm $\| \cdot \|$ and an anti-linear map $\ast : A \to A$ such that they satisfy some technical axioms and the following relation:

$$\| a^* a \| = \| a \|^2 \quad \forall a \in A$$
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$$\|a^*a\| = \|a\|^2 \quad \forall a \in A$$

**Example**

Let $X$ be a topological space. The set $C_b(X)$ of continuous and bounded functions $f : X \rightarrow \mathbb{C}$ is a C*-algebra where the product is given pointwise, the norm and the involution are defined by

$$\|f\| = \sup_{x \in X} |f(x)| \quad (f^*)(x) = \overline{f(x)}$$
Definition

Let $X$ be a locally compact Hausdorff topological space. The algebra $C_0(X)$ of functions that \textit{vanish at infinity} is the norm closure of the functions with compact support $C_c(X)$. 
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Theorem (First Gelfand-Najmark Theorem)

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**Definition**

Let $X$ be a locally compact Hausdorff topological space. The algebra $\mathcal{C}_0(X)$ of functions that **vanish at infinity** is the norm closure of the functions with compact support $\mathcal{C}_c(X)$.

**Theorem (First Gelfand-Najmark Theorem)**

- Let $X$ be a locally compact Hausdorff topological space. The space $\mathcal{C}_0(X)$ is a commutative $C^*$-algebra.

- Let $A$ be a commutative $C^*$-algebra. There exists a locally compact Hausdorff topological space $X$ such that $A \cong \mathcal{C}_0(X)$. 
The correspondence of the previous theorem is very well known:

- Points of $X$ are given by the maximal modular ideals of $A$. 
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- There is a bijective correspondence between maximal modular ideals $I \subseteq A$ and the quotient maps $\gamma : A \to A/I \cong \mathbb{C}$. Under this identification, $X$ is endowed with the weak-$*$ topology.
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- Points of $X$ are given by the maximal modular ideals of $A$.
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- The isomorphism is given by the Gelfand Transform

$$\Gamma : A \to C_0(X) \quad a \mapsto \hat{a}$$

where $\hat{a} : X \to \mathbb{C}$ is the point evaluation $\hat{a}(\gamma) = \gamma(a)$. 

How is This Possible?
Noncommutative Topological Spaces

Crucial Point

The Gelfand-Najmark correspondence is actually an equivalence of categories.

Moral of the Story

We can consider (possibly) noncommutative $C^*$-algebras as a generalization of the "usual" notion of topological space.
## Crucial Point

The Gelfand-Najmark correspondence is actually an **equivalence of categories**. In particular, this means that

- $X \simeq Y$ if and only if $C_0(X) \simeq C_0(Y)$.

- Every $C^*$-algebra homomorphism $F : C_0(Y) \to C_0(X)$ is the pullback map of a continuous (and proper) map $f : X \to Y$. 

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### Moral of the Story

We can consider (possibly) noncommutative $C^*$-algebras as a generalization of the "usual" notion of topological space.
Let $X$ be a locally compact Hausdorff topological space.

- $X$ is compact if and only if $C_0(X)$ is unital.
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- $X$ is compact if and only if $C_0(X)$ is unital.
- Given a $C^*$-algebra $A$, we define its minimal unitization $A^+$ as the smallest unital $C^*$-algebra that contains $A$ (as an essential ideal). Then

$$C_0(X)^+ \simeq C_0(X^+)$$

where $X^+ = X \cup \{\infty\}$ is the one-point compactification of $X$. 

Question: What happens if we take the maximal unitization of the $C^*$-algebra $A$?

We get the Stone-Čech compactification of $X$. 

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**Question**

*What happens if we take the maximal unitization of the $C^*$-algebra $A$? We get the Stone–Čech compactification of $X$.***
Definition

Let $A$ be a ring. An $A$-module $\mathcal{M}$ is said to be **projective** if for every surjective module morphism $\rho: \mathcal{N} \to \mathcal{M}$ there exists a module morphism $s: \mathcal{M} \to \mathcal{N}$ such that $\rho \circ s = \text{id}_{\mathcal{M}}$. 
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Theorem (Serre-Swan)

Let $X$ be a compact Hausdorff topological space.

- For every vector bundle $\pi: E \to X$, the space of sections $\Gamma(X, E)$ is a projective finitely generated $\mathcal{C}(X)$-module.
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Let $X$ be a compact Hausdorff topological space.

- For every vector bundle $\pi : E \to X$, the space of sections $\Gamma(X, E)$ is a projective finitely generated $\mathcal{C}(X)$-module.
- For every projective finitely generated $\mathcal{C}(X)$-module $M$, there exists a vector bundle $\pi : E \to X$ such that $M \cong \Gamma(X, E)$.
The topological $K$-theory of a topological space $X$ is the group of isomorphism classes of (complex) vector bundles over $X$ with addition given by the direct sum of vector bundles.

Under the identification of the Serre-Swan theorem, the NC $K$-theory group $K_0(A)$ is defined as the space of suitable equivalence classes of projections of a $C^*$-algebra $A$.

There is a correspondence between the commutative and the noncommutative $K$-theory, namely that

$$K^0(X) \simeq K_0(C_0(X))$$

for a locally compact Hausdorff topological space $X$. 
### Noncommutative Dictionary

<table>
<thead>
<tr>
<th>Topology</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous Proper Map</td>
<td>Morphism</td>
</tr>
<tr>
<td>Homeomorphism</td>
<td>Automorphism</td>
</tr>
<tr>
<td>Compact</td>
<td>Unital</td>
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<tr>
<td>1-point Compactification</td>
<td>Minimal Unitization</td>
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<tr>
<td>Stone-Čech Compactification</td>
<td>Multiplier Algebra</td>
</tr>
<tr>
<td>Open (Dense) Subset</td>
<td>(Essential) Ideal</td>
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<tr>
<td>Second Countable</td>
<td>Separable</td>
</tr>
<tr>
<td>Connected</td>
<td>Trivial Space of Projections</td>
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<tr>
<td>Vector Bundle</td>
<td>Fin. Gen. Proj. Module</td>
</tr>
<tr>
<td>Cartesian Product</td>
<td>Tensor Product</td>
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</tbody>
</table>

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Noncommutative Measure Theory
Example

Let $H$ be a Hilbert space. The space $\mathcal{L}(H)$ of linear and bounded operators $T : H \to H$ is a unital $C^*$-algebra where the product is given by the composition, the norm is

$$\| T \| = \sup_{\| x \| = 1} \| Tx \|$$

and the involution is $T \mapsto T^*$ where $T^*$ is the adjoint operator.
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Theorem (Second Gelfand-Najmark Theorem)

Any $C^*$-algebra admits a faithful representation on a suitable Hilbert space $H$. Furthermore, if $A$ is separable, $H$ can be chosen to be separable.
The **strong operator topology** on $\mathcal{L}(H)$ is the locally convex vector space topology induced by the family of seminorms

$$p_x : \mathcal{L}(H) \rightarrow [0, +\infty[,$$

$$p_x(T) = \|Tx\|$$

varying $x \in H$. 

**Von Neumann Algebras**
The strong operator topology on $\mathcal{L}(H)$ is the locally convex vector space topology induced by the family of seminorms

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A Von Neumann algebra is a SOT-closed $\ast$-subalgebra of $\mathcal{L}(H)$ for some Hilbert space $H$. 
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A **Von Neumann algebra** is a SOT-closed $\ast$-subalgebra of $\mathcal{L}(H)$ for some Hilbert space $H$.

Since every SOT-closed set is also norm closed, every Von Neumann algebra is in particular a $C^*$-algebra.
The reader should note that our definition of Von Neumann algebra is not algebraic and absolute (an algebra with axioms) but is topological and depends on having a Hilbert space.

**Theorem (The Double Commutant Theorem)**

A $C^*$-subalgebra $N \subseteq \mathcal{L}(H)$ is a Von Neumann Algebra iff:

1. $\text{id}_H \in N$
2. $N = N''$ where $N' = \{ T \in \mathcal{L}(H) \mid TS = ST \quad \forall S \in N \}$. 
Let $X$ be a compact Hausdorff topological space and $\mu$ a Radon measure on $X$. The action of $L^\infty(X, \mu)$ on the separable Hilbert space $L^2(X, \mu)$ given by multiplication makes $L^\infty(X, \mu)$ a commutative Von Neumann algebra.
Theorem

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- Let $A$ be a commutative Von Neumann algebra acting on a separable Hilbert space $H$. There exists a compact Hausdorff topological space $X$ and a Radon measure $\mu$ on $X$ such that $A \simeq L^\infty(X, \mu)$ as $C^*$-algebras.

Remark

In this form, this is not an equivalence of categories.
Theorem

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In this form, this is not an equivalence of categories.
Theorem (Riesz-Markov)

Let $X$ be a locally compact Hausdorff space.

- Given a complex Radon Measure $\mu$ on $X$, the map $I: C_0(X) \rightarrow \mathbb{C}$ given by

$$I(f) = \int_X f(x) d\mu(x)$$

is bounded and linear.
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- Given a bounded linear functional $I: C_0(X) \to \mathbb{C}$, there exists a unique complex Radon measure $\mu$ such that
  
  $$I(f) = \int_X f(x) d\mu(x) \quad \forall f \in C_0(X)$$
Open Problems

- Noncommutative Probabilities
- Noncommutative $L^p$-spaces
- Noncommutative Sobolev Spaces
- ...
Noncommutative Manifolds
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**Huge Problem**

$C^\infty(M)$ is a Frechet space: a locally convex topological vector space that is complete with respect to a translation-invariant metric. In particular its topology is generated by a countable family of seminorms that cannot be reduced to only one norm.
<table>
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<tr>
<th>Huge Problem</th>
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There’s no natural way to make $C^\infty(M)$ a $C^*$-algebra.
### Huge Problem

There’s no natural way to make $\mathcal{C}^\infty(M)$ a $C^*$-algebra.

### Crucial Point

Since $\mathcal{C}^\infty(M)$ is dense in the $C^*$-algebra $\mathcal{C}(M)$ in the sup norm, we can still follow the Gelfand-Najmark approach once we specify what in $\mathcal{C}(M)$ can be regarded as "smooth". Morally, this lack of information can be compensated by an operator $D$ that behaves like a derivative.
**Definition**

Let $\mathcal{A}$ be a unital $\ast$-algebra dense in a $C^*$-algebra $A$, $H$ an Hilbert space and $D : \text{Dom}(D) \subseteq H \to H$ a densely defined self-adjoint operator. We say that the data $(\mathcal{A}, H, D)$ is a spectral triple if

- there is a reprs. $\pi : \mathcal{A} \to \mathcal{L}(H)$ on $H$ by bounded operators
- $D$ is compatible with the action of $A$ in the sense that
  1. the resolvent operator $(D - \lambda)^{-1} : H \to H$ is compact for every $\lambda$ in the resolvent set $\rho(D)$.
  2. the action of $A$ preserves the domain of $D$ and the densely defined operator $[D, a]$ extends to a bounded operator for every $a \in \mathcal{A}$

We say that the triple is commutative if $A$ is commutative.
The definition of a spectral triple is modelled on a canonical example from the field of Spin Geometry. The next few slides will give you a glimpse on the construction of this canonical example.
Consider a finite dimensional vector space $V$ over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ endowed with a quadratic form $q: V \to \mathbb{K}$. The Clifford algebra $\text{Cl}(V, q)$ is the quotient of the tensor algebra $T(V) = \bigoplus_n V^\otimes n$ under the assumption that $v^2 = -q(v)$. The Spin groups $\text{Spin}(n)$ are defined as some (multiplicative) subgroup of the invertible elements in $\text{Cl}(\mathbb{R}^n)$. One can show that for every $n \geq 1$ the spin groups are the double coverings of $\text{SO}(n)$ in the sense that we have the SES:

$$0 \to \mathbb{Z}_2 \to \text{Spin}(n) \to \text{SO}(n) \to 0$$

Furthermore, if $n \geq 2$ they are non trivial and if $n \geq 3$ they are the universal coverings.
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Furthermore, if $n \geq 2$ they are non trivial and if $n \geq 3$ they are the universal coverings.
Let $M$ be an or. Riem. manifold. For every $p \in M$ we associate an or. orth. basis of $T_p M$: this gives a vector bundle $F$ with transition functions $g_{\alpha\beta}$. Spin($n$)

$\sigma$

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Let $M$ be an or. Riem. manifold. For every $p \in M$ we associate an or. orth. basis of $T_p M$: this gives a vector bundle $F$ with transition functions $g_{\alpha\beta}$. 

We say that $M$ is a **spin manifold** if we can find a family of functions $\tilde{g}_{\alpha\beta}$ that satisfy the cocycle conditions. In this case we denote by $\tilde{F}$ the generated bundle. This is a topological property.
Given a repres. $\gamma: \text{Spin}(n) \to \text{GL}(S)$, we define the spin bundle as $\Sigma = \tilde{F} \times_{\gamma} S$. Its sections are called spinor fields and can be regarded as $\gamma$-equivariant functions $\psi: \tilde{F} \to S$. 
Given a repres. $\gamma : \text{Spin}(n) \to \text{GL}(S)$, we define the spin bundle as $\Sigma = \tilde{F} \times_{\gamma} S$. Its sections are called **spinor fields** and can be regarded as $\gamma$-equivariant functions $\psi : \tilde{F} \to S$.

The Clifford bundle is the vector bundle on $M$ whose fibers are $\mathcal{C}l(T_pM)$ on $p \in M$. The map $\gamma$ naturally induces an action

$$\gamma : \Gamma(T^*M \otimes \Sigma) \to \Gamma(\Sigma) \quad \alpha \otimes \psi \mapsto \gamma(\alpha)\psi$$

such that $\gamma^2(\alpha) = -g(\alpha, \alpha)$.
Given a repres. \( \gamma : \text{Spin}(n) \rightarrow \text{GL}(S) \), we define the spin bundle as \( \Sigma = \tilde{F} \times_\gamma S \). Its sections are called spinor fields and can be regarded as \( \gamma \)-equivariant functions \( \psi : \tilde{F} \rightarrow S \).

The Clifford bundle is the vector bundle on \( M \) whose fibers are \( \mathcal{Cl}(T_p M) \) on \( p \in M \). The map \( \gamma \) naturally induces an action

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\]

such that \( \gamma^2(\alpha) = -g(\alpha, \alpha) \).

There exists a connection \( \nabla^S : \Gamma(M, \Sigma) \rightarrow \Gamma(M, T^* M \otimes \Sigma) \) compatible with the \( \gamma \)-action in the sense that

\[
\nabla^S(\gamma(\alpha)\psi) = \gamma(\nabla \alpha)\psi + \gamma(\alpha)\nabla^S \psi
\]
The Dirac operator \( D \) is the composition map:

\[
D : \Gamma(M, \Sigma) \xrightarrow{\nabla^S} \Gamma(M, T^* M \otimes \Sigma) \xrightarrow{\gamma} \Gamma(M, \Sigma)
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There exists an hermitian product $\langle \cdot, \cdot \rangle$ on $\Gamma(M, \Sigma)$ with respect to which $D$ is symmetric. If $M$ is complete as a metric space, $D$ is in particular essentially self-adjoint.
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We denote by $L^2(M, \Sigma)$ the Hilbert space obtained by the completion of $\Gamma(M, \Sigma)$ w.r.t $\langle \cdot, \cdot \rangle$ and by $H^1(M, \Sigma)$ the completion w.r.t $\|\psi\|_H^2 = \|\psi\|_{L^2}^2 + \|\nabla^S \psi\|_{L^2}^2$. It turns out that $D : H^1(\Sigma) \to L^2(\Sigma)$ is linear and bounded.
Using the Schroedinger-Lichnerowicz identity:

\[ D^2 = \Delta + \frac{1}{4} R \quad R \text{ scalar curvature} \]

one can prove that the resolvent operators go from \( L^2(\Sigma) \) to \( H^1(\Sigma) \) and are bounded. The Rellich-Kondrachov embedding theorem makes them compact operators.
Let $M$ be a compact orientable Riemannian spin manifold. The triple $(\mathcal{C}^\infty(M), L^2(M, \Sigma), \overline{D})$ is a commutative spectral triple that has other seven algebraic/analytic properties.
## Connes Reconstruction Principles

- **Let $M$ be a compact orientable Riemannian spin manifold. The triple $(\mathcal{C}^\infty(M), L^2(M, \Sigma), \overline{D})$ is a commutative spectral triple that has other seven algebraic/analytic properties.**

- **For every commutative spectral triple $(A, H, D)$ that satisfies the before mentioned seven properties, there exists a compact orientable Riemannian spin manifold $M$ such that $(A, H, D)$ is the canonical spectral triple as in the previous point.**
Noncommutative (Spin) Manifolds

Connes Reconstruction Principles

- Let $M$ be a compact orientable Riemannian spin manifold. The triple $(\mathcal{C}^\infty(M), L^2(M, \Sigma), \overline{D})$ is a commutative spectral triple that has other seven algebraic/analytic properties.

- For every commutative spectral triple $(A, H, D)$ that satisfies the before mentioned seven properties, there exists a compact orientable Riemannian spin manifold $M$ such that $(A, H, D)$ is the canonical spectral triple as in the previous point.

Remark

- The above mentioned association is not functorial.
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Remark

- The above mentioned association is not functorial.

- We have no idea if the theorem is true if we relax the hypothesis on compactness or spin structure.
The Dirac operator $D$ is crucial to capture the geodesic distance on the manifold by

$$d(x, y) = \sup \{|f(x) - f(y)| : f \in C^\infty(M) \text{ and } \|[D, f]\| \leq 1\}.$$
The Role of the Dirac Operator

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- the geodesic distance on the manifold by

$$d(x, y) = \sup \{|f(x) - f(y)| : f \in C^\infty(M) \text{ and } \|[D, f]\| \leq 1\}.$$  

- the dimension of the manifold.

**Theorem (Weyl’s Law)**

Let $M$ a compact connected oriented boundary-less Riemannian spin manifold of dimension $n$. Consider the positive compact operator $|D| = (D^2 + 1)^{-\frac{1}{2}}$ defined on $L^2(M, \Sigma)$ and let $\lambda_k$ be the eigenvalues listed in increasing order. Then

$$N_{|D|}(\lambda) := \# \{ k \in \mathbb{N} \mid \lambda_k \leq \lambda \}$$
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$$N_{|D|}(\lambda) := \# \{ k \in \mathbb{N} \mid \lambda_k \leq \lambda \} \sim \frac{\text{Vol}(M)}{(4\pi)^\frac{n}{2} \Gamma(\frac{n}{2} + 1)} \lambda^{\frac{n}{2}},$$

where $\Gamma$ is the Euler $\Gamma$-function.
The various additional axioms to the reconstruction theorem (that I voluntarily avoided) are related to the noncommutative formulations of:

- the orientation of the manifold (Hochschild Homology)
- the dimension of the manifold (Trace-class operators and Dixmier Ideals)
- the existence of a spin structure (Morita Equivalence of $C^*$-algebras)
- the absolutely continuity of the noncommutative integral
- ...
The Noncommutative Torus
The Commutative Torus

We want to apply the NC procedure to the torus $T^2 = S^1 \times S^1$.

First of all notice that every element $F \in C^\infty(T^2)$ can be regarded as a smooth function $f : \mathbb{R}^2 \to \mathbb{C}$ such that

$$f(x + 2\pi m, y + 2\pi n) = f(x, y) \quad \forall (m, n) \in \mathbb{Z}^2$$

via $f(x, y) = F(e^{2\pi ix}, e^{2\pi iy})$.
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By Fourier analysis, we can write $f \in C^\infty(T^2)$ as

$$f(x, y) = \sum_{(m,n) \in \mathbb{Z}^2} a_{m,n} e^{2\pi m ix} e^{2\pi n iy}$$

$$= \sum_{(m,n) \in \mathbb{Z}^2} a_{m,n} U^m V^n \quad \text{for} \ U = e^{2\pi ix}, \ V = e^{2\pi iy}$$

Note that $U^* = U^{-1}$, $V^* = V^{-1}$ and $UV = VU$. 
The Noncommutative Torus

**Definition**

The Noncommutative Torus $\mathcal{C}(T_\theta^2)$ with $\theta > 0$ is the universal $C^*$-algebra generated by two unitaries $U_1$ and $U_2$ such that

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We want now to show that from the $C^*$-algebra $\mathcal{C}(T^2_\theta)$ we can naturally build a spectral triple.
We define $L^2(T^2_\theta)$ as the completion of $C(T^2_\theta)$ with respect to the scalar product

$$\langle a, b \rangle := \tau(a^* b) \quad \text{where} \quad \tau \left( a = \sum_{n,m \in \mathbb{Z}} a_{mn} U^m V^n \right) := a_{00}$$
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The Dirac operator $D_\theta : L^2(\mathbb{T}^2_\theta) \otimes \mathbb{C}^2 \to L^2(\mathbb{T}^2_\theta) \otimes \mathbb{C}^2$ is

$$D_\theta = i(\partial_1 \otimes \sigma_1 + \partial_2 \otimes \sigma_2)$$

where $\sigma_1$ and $\sigma_2$ are Pauli matrices and $\partial_i U_j = \delta_{ij} U_j$ are commuting derivations on $C(\mathbb{T}^2_\theta)$. 
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$(C^\infty(\mathbb{T}^2_\theta), L^2(\mathbb{T}^2_\theta) \otimes \mathbb{C}^2, D_\theta)$ is a spectral triple.
Some Applications to Physics
NCG gives a unifying picture of the phase space and its observables in classical and quantum mechanics.

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**Moral of the Story**

*Heisenberg Commutation Relation = Noncommutative Torus*
Heisenberg and Tori

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Heisenberg Commutation Relation = Noncommutative Torus

Remark

In the limit \( \hbar \to 0 \) we get a commutative torus. This is not surprising if we remember Liouville-Arnold Theorem of classical mechanics.
CAN YOU SEE ??
IT'S ALL CONNECTED !
The Quantum Hall Effect

\[ J = \rho E \]

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The Classical behaviour
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The Classical behaviour

The Quantum behaviour
Let us model the Hall effect as a 2-dim lattice model with Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^2)$ and Hamiltonian $H = U + U^* + V + V^*$ where

$$(U\psi)(m, n) = \psi(m - 1, n)$$

$$(V\psi)(m, n) = e^{-2\pi i \phi m} \psi(m, n - 1)$$

and $\phi$ is interpreted as the magnetic flux through a unit cell.
Note that $UV = e^{-2\pi i\phi} VU$ so that $C^*(U, V)$ is the noncommutative torus $A_{-\phi} = C(T^2_{-\phi})$.

Provided that the Fermi level $\mu \notin \sigma(H)$ (that is, the system is an insulator), the Fermi projection $P_\mu$ defines a class $[P_\mu]$ in the $K$-theory group $K_0(A_\phi)$.

Let $X_j$ be the position operators $(X_j\psi)(n_1, n_2) = n_j\psi(n_1, n_2)$. One can show that

$$\left( C(T^2_\phi), \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2, D = \begin{pmatrix} 0 & X_1 - iX_2 \\ X_1 + iX_2 & 0 \end{pmatrix} \right)$$

is a spectral triple. Up to "homotopy" this defines a class $[X]$ in $K$-homology.
There is a perfect pairing $K_0(A_\phi) \times K^0(A_\phi) \to \mathbb{Z}$ between $K$-theory and $K$-homology given by the Fredholm index

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**Theorem (Kubo-Bellisard Formula)**

$$\rho_{11} = \frac{e^2}{\hbar} \text{Index}(P_\mu(X_1 + iX_2)P_\mu)$$
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**Moral of the Story**

*Noncommutativity $\implies$ Quantization of the Current*
A Short Bibliography


